TEMPERATURE GRADIENT SENSITIVITY
OF THE FAME BASIC ANGLE

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ABSTRACT

If the compound mirror assembly (CMA) in FAME were to be fabricated of a single block of Zerodur (instead of silicon carbide with glued-on Zerodur wedges as currently proposed), then, assuming the thermal environment indicated in the Step 1 proposal, changes in the basic angle due to temperature gradient fluctuations during the course of one spacecraft spin period are confined to ~35 μas. This gives rise to the interesting possibility that the proposed laser metrology system, whose sole functional purpose is to measure short-term changes in the basic angle, may not be needed if further, possibly inexpensive, attention is given to the thermal environment of the CMA. A tolerance of 25 μas on changes in the basic angle can be met passively if short-term gradient changes are on the order ~5 mK/m or less.

Additionally, there is a potential problem that is independent of the metrology question: warping of the CMA mirrors alone due to static gradients (as illustrated in the Step 1 proposal) will introduce wavefront errors of order ~18 nm (~λ/30at λ = 550 nm). Similar warping of the primary mirror will also occur.

Key words: FAME — basic angle — thermal gradients

1. INTRODUCTION AND SUMMARY OF RESULTS.

This memo addresses the temperature gradient sensitivity of the FAME basic angle. I adopt a simple analytical model and assume that the so-called "compound mirror assembly" (CMA) is formed from a contiguous block of glass,[1] or perhaps two bonded glass slabs (Figure 1). I consider for analytical simplicity two cases: "longitudinal" gradients, where the temperature gradient is in the direction from the CMA to the primary (parallel to the primary mirror axis of symmetry), and "transverse" gradients, which are perpendicular to the longitudinal gradients. Both cases assume, as a worst-case scenario, that the gradient direction lies in the plane of the optical bench.

For convenience, I summarize here the calculations that follow in Sections 2 and 3. I find that the change in basic angle ψ due to the imposition of a gradient β is

$$\delta\psi = \frac{-2a}{\cos^2\chi}\,\gamma_\parallel + O(\gamma_\parallel^2)$$

for the longitudinal case and

$$\delta\psi = 2\frac{h + a\tan\chi}{\cos^2\chi}\,\gamma_\perp + O(\gamma_\perp^2)$$

for the transverse case. Here, $\gamma \equiv \alpha \cdot \beta$, where α is the CTE of the material, χ is the angle of the CMA mirrors with respect to the primary mirror symmetry axis, *h* is the distance along the symmetry axis from the CMA support(s) to the CMA mirrors, and *2a* is the width of the CMA block (same as the primary mirror width, currently 60 cm). (cf. Figures 1, 3, and 6.) These equations impose the temperature gradient constraints

$$\beta_\parallel \le \frac{\cos^2\chi}{2a\,\alpha}\,\tau$$

and

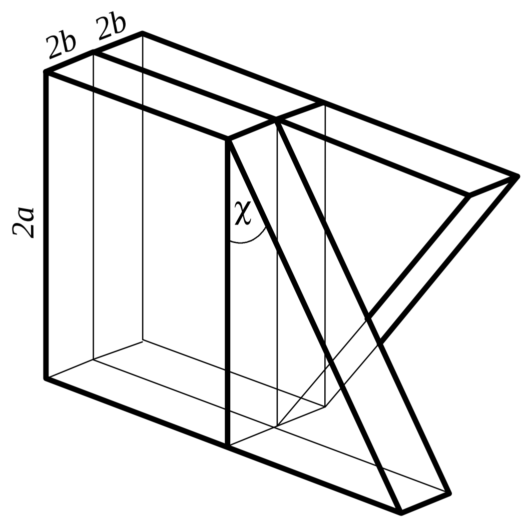

FIG. 1. — Diagram of FAME compound mirror assembly.

$$\beta_\perp \le \frac{\cos^2\chi}{2\alpha(h + a\tan\chi)}\,\tau$$

where τ is the allowed tolerance on δψ. For h = 0, χ = 45 deg, τ = 25 μas, a = 30 cm, and α = $2 \cdot 10^{-8}\,K^{-1}$, we have $\beta_\parallel \le 5.1$ mK/m and $\beta_\perp \le 5.1$ mK/m . Changes in the temperature gradients within the CMA of this order or larger on timescales of the order of the spacecraft spin period or less will require direct means of either controlling or measuring the basic angle fluctuations. (Static gradients[2] are of little or no concern in this respect, since the basic angle is a solution parameter in the data reduction process.) Figure 2 is a contour plot of the temperature gradient tolerance in mK/m as a function of δψ in μas and the CTE in units of 10^{-8} K^{-1}. The CTE of the ultra-low

[1] Current design calls for the CMA to consist of silicon carbide with thin Zerodur mirrors attached with glue. However, the CTE of SiC is ~200 times that of Zerodur.
[2] I.e., gradients that do not change on timescales smaller than the spacecraft spin period.

expansion glass Zerodur is at most $5x10^{-8}$ K^{-1} in the temperature range 0-50 C. For smaller individual pieces (less than ~600 lb.) it is not unreasonable to expect values that are better than this. I use $2x10^{-8}$ K^{-1} in this memo for illustrative purposes. For comparison, the CTE of silicon carbide is $4.3x10^{-6}$ K^{-1}.

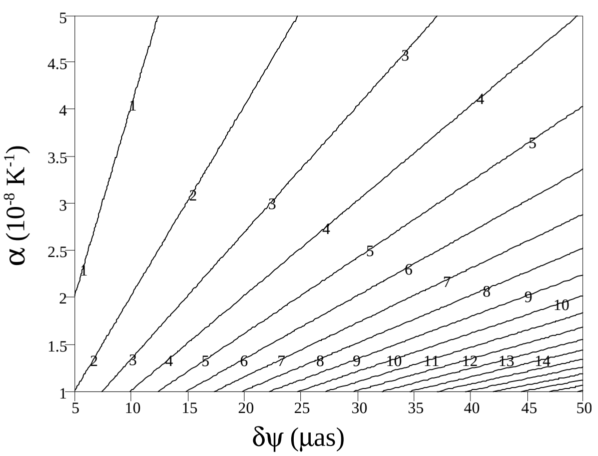

FIG. 2. — Contours of temperature gradient tolerance in mK/m vs. CTE and change in the basic angle.

Initial thermal work performed at JPL for the FAME project indicates that the thermal environment may in fact be stable to better than ~7 mK changes over one spin period.[3] If this is the case, and if further analysis supports the results found here, then perhaps relatively inexpensive further attention to the thermal environment of the CMA in particular could replace the more costly, complex, and untested (in space) laser metrology system currently proposed.

Additionally, it should be pointed out that, since the CMA lies in collimated light beams, the additional longitudinal shifts from non-ideal placement of the CMA attachment points to the optical bench have no significant effect for longitudinal gradients. However, for transverse gradients the deformation of the glass between the supports and the mirrors contributes to the rotation of each mirror in opposite directions. Therefore, it is important that *h* be as close as possible to zero (Figures 3 and 6).

Finally, there is a potential problem concerning beam divergence due to curling of the mirrors from static gradients.[4] Changes in the temperature gradients are likely to be too small to have a noticeable effect. Wavefront errors on the order of 18 picometers would result from gradient changes of 5 mK/m. However, *static* thermal gradients, according to a JPL study,[5] are likely to be ~5 K/m. This would produce a noticeable warpage of the mirrors, with associated wavefront error of order ~30 nm, or ~λ/18 at λ = 550 nm.

2. LONGITUDINAL GRADIENT CASE.

2.1. Derivation of the Surface Perturbation.

Consider a cylindrical coordinate system $(z,\rho,\theta)^{\top}$ embedded in a homogeneous medium whose coefficient of thermal expansion is α °K^{-1}. For a linear temperature gradient β = dT/dz along the z coordinate axis (Figure 3), the perturbation of the position of a point in the medium is given by

$$\vec{u} = \alpha\beta \begin{pmatrix} \frac{1}{2}(z^2-\rho^2) \\ z\rho \\ 0 \end{pmatrix} \tag{1}$$

(R.D. Reasenberg, private communication). In the u_z term, $\frac{1}{2}z^2$ arises from linear expansion of the material due to the gradient, while the $-\frac{1}{2}\rho^2$ component is due to stresses within the material set up by the gradient. The latter term is the source of the familiar curling effect of an initially flat disk. The $u_\rho = \alpha\beta z\rho$ term is just the linear expansion of the material, at a height z, in the ρ direction, due to the cumulative effects of a gradient in the z direction. Thus, for a plane inclined to the direction of the gradient by an angle χ, the perturbed position on the initially planar surface as a function of the unperturbed coordinates is

$$\vec{r}' = \begin{pmatrix} z' \\ \rho' \\ \theta' \end{pmatrix} = \vec{r} + \vec{u} = \begin{pmatrix} s+\frac{1}{2}\alpha\beta(s^2-\rho^2) \\ \rho(1+\alpha\beta s) \\ \theta \end{pmatrix} \tag{2}$$

where $x=\rho\cos\theta$, $y=\rho\sin\theta$, and $s=h+\rho\cos\theta\tan\chi$ is the thickness of the material from the CMA support(s) along the gradient (Figure 3). In the coordinate system shown in Figure 3, if the attachment points of the CMA to the optical block are located at z=0, then h=0. More generally, for attachment at a level z = z_0, then h = -z_0.

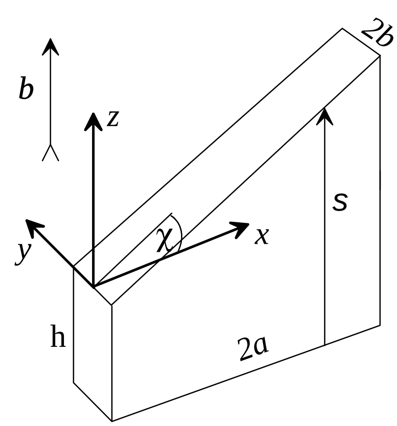

FIG. 3. — One of the CMA wedges, illustrating the coordinate system for longitudinal gradients.

Equation (2) contains a mix of perturbed and unperturbed coordinates and is therefore not useful in its current form. We

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3 The ~7 mK/m fluctuation shown in the Step 1 proposal is actually a worst-case scenario which was driven by Earthlight at perigee (J. McGuire, private communication).

4 The effects of gravity unloading and non-uniform CTE should also be looked at in this respect.

5 Results are shown in the Step 1 proposal (see Figure 2.2-8 on page 19).

wish to express the perturbed surface in terms of the perturbed coordinates. From the ρ' component of (2), we have

$$\rho' = \rho\,[1 + \alpha\,\beta\,(h + \rho\cos\theta\tan\chi)] \tag{3}$$

Solving for ρ as a function of ρ', we find

$$\rho = \frac{\sqrt{(1+\gamma h)^2 + 4\gamma\rho'\cos\theta\tan\chi} - (1+\gamma h)}{2\gamma\cos\theta\tan\chi} \tag{4}$$

where $\gamma \equiv \alpha\cdot\beta$. The coefficient of thermal expansion is a small quantity, so we can expand on γ:

$$\rho = \rho' - (h + \rho'\cos\theta\tan\chi)\rho'\gamma + O(\gamma^2) \tag{5}$$

Use (5) for ρ in the z' component of (2), expanding on γ, to get

$$\begin{aligned} s' &= s + \tfrac{1}{2}(s^2 - \rho^2)\gamma \\ &= (h + \rho'\cos\theta\tan\chi) \\ &\quad -[(1+\cos^2\theta\tan^2\chi)\rho'^2 - h^2]\,\gamma \; + O(\gamma^2) \end{aligned} \tag{6}$$

Switching back to Cartesian coordinates and dropping the primes, we have the equation for the perturbed glass thickness:

$$s = h + x\,\tan\chi + \Delta s \tag{7}$$

where

$$\Delta s = -\frac{1}{2}\left(\frac{x^2}{\cos^2\chi} + y^2 - h^2\right)\gamma + O(\gamma^2) \tag{8}$$

Notice that $x/\cos\chi$ is the position along the mirror surface, which is physically of length $2a/\cos\chi$.

A positive gradient along the z axis produces a warp in both the x and y directions. Relative to x=0, the far end of the mirror at x=2*a* sags along x in the shape of a parabola. Along the y dimension (x=0), we also have a downward-sagging parabolic warp.

2.2. A Numerical Example.

The warps introduced by a positive temperature gradient of 10 mK/m with $\alpha = 2\cdot10^{-8}K^{-1}$ are illustrated in Figures 4 and 5. In Figure 4, the surface perturbation units are nanometers. The warp across the short dimension (Figure 5, on the next page) is shown in picometers.

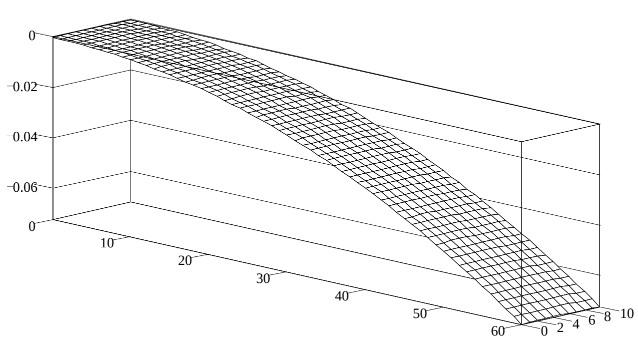

FIG. 4. — Surface perturbation Δs (nm) for a 10 mK/m longitudinal temperature gradient. The warp across the short dimension is two and a half orders of magnitude smaller than the warp across the long dimension.

2.3. Resulting Change in the CMA Basic Angle. Constraint on Gradient Magnitude.

The longitudinal excursion of the end of the mirror (x=2a) relative to the beginning (x=0) is

$$\Delta s(\gamma, y = h = 0, x = 2\,a) = \frac{-2\,a^2}{\cos^2\chi}\gamma + O(\gamma^2) \tag{9}$$

This corresponds to an angle

$$\frac{\delta\psi}{2} = \frac{\Delta s}{2\,a} = \frac{-a}{\cos^2\chi}\gamma + O(\gamma^2) \tag{10}$$

The change in the basic angle ψ is approximately twice this amount. For a tolerance τ such that $\delta\psi \le \tau$, the constraint on the longitudinal temperature gradient is then

$$\beta \le \frac{\cos^2\chi}{2\,a\,\alpha}\tau + O(\tau^2) \tag{11}$$

For the specific case $\chi = 45\deg$, $\tau = 25\,\mu rad$, $a = 30\,cm$, and $\alpha = 2\cdot10^{-8}K^{-1}$, we have

$$\frac{\cos^2\chi}{2\,a\,\alpha}\tau = 5.1\,\frac{\mathrm{mK}}{\mathrm{m}}$$

A more careful way to determine the change in angle is to average Δs across the mirror by integration. The slope of the tangent along the x direction is

$$\frac{\partial\Delta s(x,y)}{\partial x} = \frac{-x}{\cos^2\chi}\gamma + O(\gamma^2) \tag{12}$$

Hence the change in angle averaged over the 2*a* by 2*b* rectangular surface is

$$\tan\frac{\delta\psi}{2} \approx \frac{\delta\psi}{2} = \frac{1}{4\,a\,b}\int_0^{2a}\int_{-b}^{b} w(x,y)\frac{\partial\Delta s(x,y)}{\partial x}\,dy\,dx \tag{13}$$

where w(x,y) is a weighting function. Evaluating the integral with w(x,y) = 1, we find

$$\frac{\delta\psi}{2} = \frac{-a}{\cos^2\chi}\gamma + O(\gamma^2) \tag{14}$$

To first order we recover the approximate result, eq. (10). The mirror width 2*b* only enters in at second order and is therefore negligible. Solving for γ and imposing a basic angle tolerance τ, we have that the temperature gradient constraint is

$$\beta \le \frac{\cos^2\chi}{2\,a\,\alpha}\tau \; + O(\tau^2) \tag{15}$$

We again recover the first-order approximation, this time in eq. (11). So, for all practical purposes, the first-order approximation is adequate.

Notice that there is no dependence of δψ on the distance *h* (cf. Figures 3 and 6). This is because, as shown by eq. (8), a longitudinal gradient in a homogeneous medium causes an even longitudinal displacement of a planar surface that is perpendicular to the gradient direction. The term involving *h* is of the form $\Delta s = \frac{1}{2}h^2\gamma$, which is independent of position perpendicular to the z axis. Hence it is only the material between z=0 and the mirror surface that contributes to a rotation of the basic angle. This would seem to indicate that placement of the supports between the CMA and the optical bench is unimportant, at least for longitudinal temperature gradients. However, such

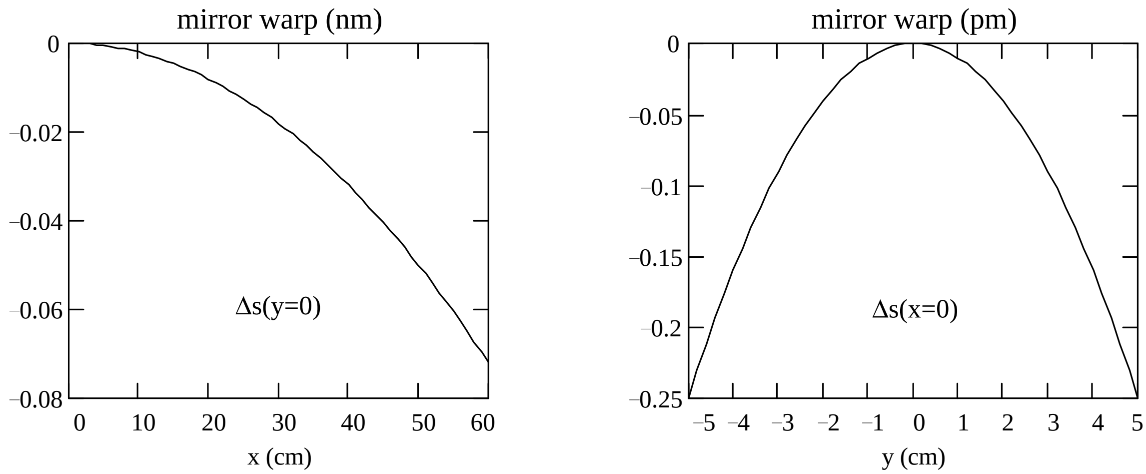

FIG. 5. — Surface perturbation slices along the long (left) and short (right) dimensions.

is not the case for transverse gradients, as we shall see in Section 3.

2.4. Beam Divergence.

Curvature of the mirrors introduced by the temperature gradient will destroy the collimation of the input beam upon reflection. In general, the radius of curvature of a function g(x) is

$$R = \left(\frac{d\theta}{d\Sigma}\right)^{-1} = \frac{\left[1 + \left(\frac{dg}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2g}{dx^2}} \equiv 2f \tag{16}$$

where θ is the angle of the tangent line at g(x), Σ is arc length along the curve, and f is the equivalent focal length. Using Δs from eq. (8) in eq. (16) yields

$$f_a = \frac{-\cos^2\chi}{2\gamma} \quad \text{and} \quad f_b = -\frac{1}{2\gamma} \tag{17}$$

where f_a and f_b are the focal lengths along the long (2a) and short (2b) dimensions. A CTE of $\alpha = 2 \cdot 10^{-8} K^{-1}$ and a temperature gradient $\boldsymbol{b}$ = 5 mK/m produce focal lengths of 2.5 and 5.0 gigameters. At the edge of the beam launched toward the primary (ρ=30 cm), the wavefront from such a mirror would lag the wavefront center by about 18 pm. This represents an insignificant wavefront error. However, static thermal gradients are on the order 5 K/m. Such gradients along the CMA would produce a wavefront error of order ~18 nm, or ~λ/30 at λ = 550 nm. The optical design goal is to achieve a wavefront error of 4 nm. It appears that the likely thermal gradients will cause a wavefront error many times this large.

3. TRANSVERSE GRADIENT CASE.

3.1. Derivation of the Surface Perturbation.

Recall eq. (1):

$$\vec{\boldsymbol{u}} = \alpha\beta \begin{pmatrix} \frac{1}{2}(z^2 - \rho^2) \\ z\rho \\ 0 \end{pmatrix} \tag{18}$$

For the transverse case, we rotate the coordinates so that the z axis is again parallel to the temperature gradient (Figure 6). Eq. (18) then applies without change. The unperturbed surface is now expressed in the form

$$s = h + (2a - z)\tan\chi = \rho\cos\theta \tag{19}$$

so that the mirror plane is described by the constraint equation

$$z = 2a + \frac{h - \rho\cos\theta}{\tan\chi} \tag{20}$$

Hence the perturbation at the surface is

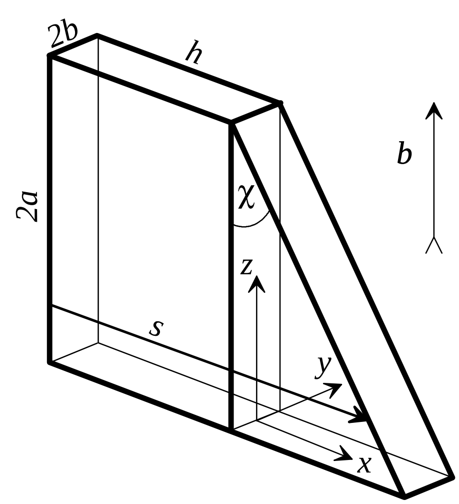

FIG. 6. — One of the CMA wedges, illustrating the coordinate system for transverse gradients.

$$\vec{\boldsymbol{u}} = \gamma \begin{bmatrix} \frac{1}{2}\left(2a + \dfrac{h-\rho\cos\theta}{\tan\chi}\right)^2 - \frac{1}{2}\rho^2 \\ 2\,a\,\rho + \dfrac{h-\rho\cos\theta}{\tan\chi}\rho \\ 0 \end{bmatrix} \tag{21}$$

so that the perturbed surface has the form

$$\begin{aligned} \vec{\boldsymbol{r}}' &= \vec{\boldsymbol{r}} + \vec{\boldsymbol{u}} \\ &= \begin{bmatrix} \left(2a + \frac{h-\rho\cos\theta}{\tan\chi}\right) + \frac{1}{2}\left[\left(2a + \frac{h-\rho\cos\theta}{\tan\chi}\right)^2 - \rho^2\right]\gamma \\ \rho\left[1 + \left(2a + \frac{h-\rho\cos\theta}{\tan\chi}\right)\gamma\right] \\ \theta \end{bmatrix} \end{aligned} \tag{22}$$

This time the independent variable of interest is z', so we would like to express the radius ρ in terms of the perturbed "height," z', and use that in the z' component of (22). From the ρ' component of (22), we have

$$\rho' = \rho + \left(2a + \frac{h-\rho\cos\theta}{\tan\chi}\right)\rho\,\gamma \tag{23}$$

Solving for ρ, we find

$$\rho = \rho' - \left(2a + \frac{h-\rho'\cos\theta}{\tan\chi}\right)\rho'\gamma + O(\gamma^2) \tag{24}$$

Hence, dropping the primed notation, the z component in the perturbed coordinates becomes

$$\begin{aligned} z \;=\; & \left(2a + \frac{h-\rho\cos\theta}{\tan\chi}\right) \\ & + \frac{1}{2}\left[4a^2 - \left(1 + \frac{\cos^2\theta}{\tan^2\chi}\right)\rho^2 + \frac{h+4a\tan\chi}{\tan^2\chi}h\right]\cdot\gamma \\ & + O(\gamma^2) \end{aligned} \tag{25}$$

Invert this to get ρ as a function of z. We find

$$\begin{aligned} \rho \;=\; & \frac{h+(2a-z)\tan\chi}{\cos\theta} \\ & + \frac{\gamma}{2}\left[\frac{2h+(4a-z)\tan\chi}{\cos\theta}\,z - \frac{(h+(2a-z)\tan\chi)^2}{\cos^3\theta}\tan\chi\right] \\ & + O(\gamma^2) \end{aligned} \tag{26}$$

Convert to Cartesian coordinates to obtain

$$\begin{aligned} x \;=\; & h + (2a-z)\tan\chi \\ & + \tfrac{1}{2}[(2h+(4a-z)\tan\chi)\,z \\ & \quad - \tfrac{x^2+y^2}{x^2}(h+(2a-z)\tan\chi)^2\tan\chi\,]\cdot\gamma \\ & + O(\gamma^2) \end{aligned} \tag{27}$$

Solve (27) for x. We find

$$x = h + (2a-z)\tan\chi + \Delta s \tag{28}$$

where Δs is given below. Hence the perturbed surface height in the case of a transverse temperature gradient is

$$s = h + (2a-z)\tan\chi + \Delta s \tag{29}$$

where

$$\begin{aligned} \Delta s \;=\; & \Big\{z\,h + \frac{1}{2}[z(4a-z) - h^2 - y^2]\tan\chi \\ & - \frac{1}{2}(2a-z)(2h+(2a-z)\tan\chi)\tan^2\chi\Big\}\cdot\gamma \\ & + O(\gamma^2) \end{aligned} \tag{30}$$

3.2. Resulting Change in the CMA Basic Angle. Constraint on Gradient Magnitude.

Since the two wedge mirrors are tilted in opposite directions (Figure 1), a positive temperature gradient across one mirror is a negative gradient across the other, in terms of the effect on the tilt angle. Therefore the change to the basic angle will again be twice the change in tilt of one of the mirrors:

$$\frac{\delta\psi}{2} = \frac{\Delta s(z,y)}{2a} \tag{31}$$

Now integrate Δs across the mirror. The slope of the tangent along the z direction is

$$\frac{\partial\Delta s(z,y)}{\partial z} = \frac{h+(2a-z)\tan\chi}{\cos^2\chi}\gamma + O(\gamma^2) \tag{32}$$

The change in angle, averaged over the 2*a* by 2*b* plane, is then

$$\frac{\delta\psi}{2} = \frac{1}{4\,a\,b}\int_0^{2a}\int_{-b}^{b} w(z,y)\frac{\partial\Delta s(z,y)}{\partial z}\,dy\,dz \tag{33}$$

Evaluating (33) with w(z,y) = 1, we have simply

$$\frac{\delta\psi}{2} = \frac{h+a\tan\chi}{\cos^2\chi}\gamma + O(\gamma^2) \tag{34}$$

Solving for γ, we find that the temperature gradient constraint is

$$\beta \le \frac{\cos^2\chi}{2\,\alpha\,(h+a\tan\chi)}\tau + O(\tau^2) \tag{35}$$

For the case h = 0, $\chi = 45$ deg , $\tau = 25\,\mu rad$, $a = 30\,cm$, and $\alpha = 2\cdot 10^{-8}K^{-1}$, we have

$$\frac{\cos^2\chi}{2\,a\,\alpha\,\tan\chi}\tau = 5.1\,\frac{\mathrm{mK}}{\mathrm{m}}$$

The tolerance on the transverse temperature gradient due to a constraint on the basic angle deviation is similar to that of the longitudinal gradient, differing by a factor $\tan\chi$ and, more importantly, by a dependence on the distance *h*.

3.3. Beam Divergence.

The curvature of the mirrors introduced by a transverse temperature gradient will cause the collimated input beam to diverge. Using eq. (16) to calculate the radius of curvature for the surface function given by eq. (30), we find

$$f_a = \frac{-\cos^2\chi}{2\gamma\tan\chi} \quad \text{and} \quad f_b = -\frac{1}{2\gamma\tan\chi} \tag{36}$$

where f_a and f_b are again the focal lengths along the long and short dimensions. Note the similarities to eqs. (17). A CTE of $\alpha = 2\cdot 10^{-8}K^{-1}$ and a temperature gradient b = 5 mK/m again produce focal lengths of 2.5 and 5.0 gigameters, since I'm using χ = 45 deg. The conclusions of section 2.4 also hold for the transverse case.